Analysis of Batch Arrival Queue with Variant Working Vacations, Server Breakdowns and Retention of Reneged Customers

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Abstract- This paper investigates an infinite buffer batch arrival single server queueing system with variant working vacations and server breakdowns wherein customers arrive according to a Poisson process. The server may breakdown during regular busy period. During working vacations, the customer may renge due to impatience and retention policy is adopted to retain the reneging customers. It is assumed that service times, vacation times, repair times and reneging times are all exponentially distributed and are all mutually independent. We derive the probability generating function of steady state probabilities and obtain performance measures of the system. In addition, cost model is formulated to determine the optimal service rate during busy period using quadratic fit search method.

Keywords- Batch Arrival Queue, Working Vacations, Server Breakdowns, Probability Generating Function, Reneging, Retention.

I. INTRODUCTION

During the past few years, there has been an increasing interest in studying queueing systems. Queueing models in which server is unavailable for occasional intervals of times are called vacation queues. Queueing are effective tools in modelling and analysis of various complex systems, such as computer systems, telecommunication systems, flexible manufacturing systems and service systems. Readers are referred to [3], [13],[14] and [6] for excellent treatment of various types of vacation queues. Servi and Finn [12] introduced the concept of multiple working vacations with the study of M/M/1/WV queue. [18] generalized [12] M/M/1/WV queue to an M/G/1/WV queue. [20] analyzed an M/M/1 queuing system with impatient customers and variant of multiple working vacation policy.

Queueing models with bulk input have enormous applications in computer networks and communication systems where the units arrive in batches. For the batch arrival queues, [19] investigated a bulk input MX/M/1 queue with single working vacation. The probability generating function of the stationary system length distribution is derived using matrix analytic method. The stochastic decomposition structure of the system length has been derived which shows the relationship with that of MX/M/1 queue without vacation. A similar analysis has been carried out in [2] for MX/M/1 queue with MWV and [11] studied an MX/M/1 queue with working breakdown.

In classical queueing systems, it is often assumed that the service station is reliable. However, we meet a situation where the server may fail and sent for repair in practice. [4]introduced queueing models with server breakdowns. They investigated a multiple vacation queueing model, where service station is subjected to breakdown while in operation. [17] developed exact steady state solutions of an N-policy M/G/1 queueing system with server breakdowns.

In real life, many queueing situations arise wherein customers are discouraged by long queue and may leave the queue without getting the service. The notion of customer impatience appeared in the queueing theory in the work of [5]. [1] presented the analysis of M/M/1, M/G/1 and M/M/c queueing models with reneging. [7] analyzed a multi-server batch arrival MX/M/c queue with impatient customers. [15] analyzed variant vacation queue with balk ing and reneging. [16] analyzed variant vacation on batch arrival queue with reneging and server breakdowns.

Queueing theory has successfully been applied to various congestion situations involving revenue generations through servicing customers. Customer impatience has become the huge problem of all types of enterprises. They are constantly working towards customer retention for the better future prospects. Thus a reneged customer can be convinced by providing certain convincing mechanism to stay in the queue for the completion of his service. [8] analyzed an M/M/1/N queueing system with retention of reneged customers and balk ing. They also analysed [9] an M/M/1 feedback queueing system with retention of reneged customers and balk ing. Batch arrivals are one of the important concepts of queueing theory, which represents many real life situations closely. Due to batch arrivals and server vacations, there may be reneging which can affect the system to a great extent. In view of this, we have considered the work of [8] with bulk input, variant working vacations and server breakdowns. We have analysed the model using probability generating function method and obtained the closed form relations for the steady state.
probabilities. Further some performance measures are derived and their affect due to some variations in model parameters and sensitivity analysis is carried out through numerical investigations. Cost optimization is also carried out using Quadratic fit search method.

The rest of the paper is organized as follows. Sections 2 and 3 present the model description and mathematical formulation respectively. In Section 4 performance measures and cost model are given. Some numerical results are discussed in Section 5 in the form of tables and graphs. Section 6 concludes our paper.

II. MODEL DESCRIPTION

In this paper, we consider an MX/M/1 queueing system with variant working vacations, server breakdowns and customer retention. Customers arrive in batches according to a Poisson process with rate. The arrival batch size is a random variable with probability mass function. The service is provided by single server with exponential service rate. The server is subject to breakdown during busy periods with exponential breakdown rate Whenever the server fails it is immediately sent to a repair facility, where repair times are exponentially distributed with rate.

A customer who arrives and finds the server busy or broken-down has to wait in the queue until service is available. In the case the server breaks down when the service is in progress, it is sent for repair and the customer who has just being served must wait for the server to come back and complete his remaining service. At the end of a service, if there is no customer in the system, the server begins working vacation of random length which is exponentially distributed with parameter $\phi$. During WV service is provided according to a Poisson distribution with parameter $\xi$.

If the server finds customer at a WV completion instant, it returns to a regular busy period; otherwise, the server takes WV’s sequentially until $k$ consecutive WV’s are complete; after which server switches to normal busy period staying idle or busy depending on the number of the customers in the system. During WV, customers becomes impatient and they may renge from the queue. Reneging times are exponentially distributed with parameter $\alpha$. The average reneging rate of a customer is given by $\alpha_n = (n-1)\alpha, n \geq 1$, where $n$ denotes number of customers in the system. An impatient customer can be made to stay in the service system for his service by utilizing certain convincing mechanism. Such customers are termed as retained customers. When a customer gets impatient he may leave the queue with probability say $p_2$ or may stay back in the queue with probability $q_1 = 1 - p_2$.

III. MATHEMATICAL FORMULATION OF THE MODEL

Let at time $t$, $L(t)$ be the number of customers in the system and $J(t)$ be the state of the server which is defined as

$$J(t) = \begin{cases} j, & \text{the server is on the } (j+1)^{\text{th}} \text{ WV for } j = 0, 1, \ldots, k - 1, \\ k, & \text{the server is idle or busy.} \end{cases}$$

The process $\{L(t), J(t), t \geq 0\}$ defines a continuous-time Markov process with state space $\mathcal{X} = \{(n,j): n \geq 0, j = 0, 1, \ldots, k, \beta d\}$. Let $P_{n,j} = \lim_{t \to \infty} P(L(t) = n, J(t) = j), n \geq 0, j = 0, 1, \ldots, k$ and $j = \beta d$ denote the steady state probabilities of the process $\{L(t), J(t), t \geq 0\}$.

Using Markov theory, the set of balance equations at steady state are given by

$$\begin{align*}
(\lambda + \phi)P_{0,0} &= \xi P_{1,0} + \mu P_{1,k}, \\
(\lambda + \phi + \xi)P_{1,0} &= \lambda b_1 P_{0,0} + (\xi + \alpha P_1)P_{2,0}, \\
(\lambda + \phi + \xi + (n-1)\alpha P_1)P_{n,0} &= \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (\xi + n\alpha P_1)P_{n+1,0}, n \geq 2, \\
(\lambda + \phi)P_{0,j} &= \xi P_{1,j} + \phi P_{0,j-1}, 1 \leq j \leq k - 1, \\
(\lambda + \phi + \xi)P_{1,j} &= \lambda b_1 P_{0,j} + \xi P_{2,j} + \alpha P_{2,j}, 1 \leq j \leq k - 1, \\
(\lambda + \phi + \xi + (n-1)\alpha P_1)P_{n,j} &= \lambda \sum_{m=1}^{n} b_m P_{n-m,j} + (\xi + n\alpha P_1)P_{n+1,j}, n \geq 2, 1 \leq j \leq k - 1, \\
\lambda P_{0,k} &= \phi P_{0,k-1}, \\
(\lambda + \mu + \beta)P_{1,k} &= \lambda b_1 P_{0,k} + \mu P_{2,k} + \phi \sum_{j=0}^{k-1} P_{1,j} + \gamma P_{1,\beta d},
\end{align*}$$

where $b_n$ denotes the batch size.
\( (\lambda + \mu + \beta) P_{n,k} = \lambda \sum_{m=1}^{n} P_{n-m,k} + \mu P_{n+1,k} + \gamma P_{n,b,d} + \phi \sum_{j=0}^{n-1} P_{n,j}, \quad n \geq 2, \)  
\( (\lambda + \gamma) P_{1,b,d} = \beta P_{1,k}, \)  
\( (\lambda + \gamma) P_{n,b,d} = \lambda \sum_{m=1}^{n-1} P_{n-m,b,d} + \beta P_{n,k}, \quad n \geq 2, \)  
And the normalizing condition is  
\( \sum_{n=0}^{\infty} \sum_{j=0}^{n} P_{n,j} + \sum_{n=1}^{\infty} P_{n,b,d} = 1. \)  
Define the probability generating functions as  
\( G_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n, \quad 0 \leq z \leq 1, \quad j = 0, 1, 2, \ldots, k. \)  
And \( G_{b,d}(z) = \sum_{n=1}^{\infty} P_{n,b,d} z^n. \)  
Probability generating function of arrival batch size \( X \) is  
\( G(z) = \sum_{i=1}^{\infty} b_i z^i, \quad |z| \leq 1 \) and \( G(1) = \sum_{i=1}^{\infty} b_i = 1. \)  
We assume that arrival batch size \( (X) \) follows geometric distribution with parameter \( \theta. \)  
\( P(X = l) = (1 - \theta) \theta^{l-1} q, \quad 0 < q < 1 \) \( (l = 1, 2, \ldots) \)  
It is observed that  
\( G(z) = \frac{qz}{1-(1-q)z}, \quad G'(1) = \frac{1}{q} \) and \( G''(1) + 2G'(1) = \frac{2}{q} z. \)  
Multiplying equations (1), (2) and (3) by \( z^n \) and summing over all possible values of \( n, \) we obtain  
\( \alpha p_1 z (1 - z) G_0'(z) - [\lambda z (1 - G(z)) + (\alpha p_1 - \xi) (1 - z) + \phi z] G_0(z) = (\xi - \alpha p_1) (1 - z) P_{0,0} - \mu z P_{1,k}. \)  
Similarly from equations (4), (5), (6) and (7), (8), (9) we obtain, respectively  
\( \alpha p_2 z (1 - z) G_1'(z) - [\lambda z (1 - G(z)) + (\alpha p_2 - \xi) (1 - z) + \phi z] G_1(z) = (\xi - \alpha p_1) (1 - z) P_{0,j} - \phi z P_{0,j-1}. \)  
Taking \( z = 1 \) in equations (17) and (18), we obtain  
\( \phi G_0'(1) = \mu P_{1,k} \) and \( G_j(1) = P_{0,j-1}, \) \( 1 \leq j \leq k - 1. \)  
Multiplying equation (21) from both sides by integrating factor \( I.F. = (1 - (1 - q)z) \alpha p_1 z \)  
We obtain  
\( \frac{\lambda}{\lambda - \alpha p_1} K_0(z) = \phi \frac{\phi}{\alpha p_1} \left[ \frac{G_0(z)}{\lambda \alpha p_1} - \frac{\mu}{\alpha p_1} K_0(z) P_{0,k} \right], \)  
Where \( K_0(z) = \int_0^{\lambda} (1 - (1 - q)x) \frac{\phi}{\alpha p_1} \left[ \frac{G_0(z)}{\lambda \alpha p_1} - \frac{\mu}{\alpha p_1} K_0(z) P_{0,k} \right] \)  
\( K_1(z) = \int_0^{\lambda} (1 - (1 - q)x) \phi \left[ \frac{G_0(z)}{\lambda \alpha p_1} - \frac{\mu}{\alpha p_1} K_0(z) P_{0,k} \right] \)  
Proceeding similarly, equation (22) gives  
\( G_j(z) = \frac{\phi}{\alpha p_1} \left[ \frac{G_0(z)}{\lambda \alpha p_1} - \frac{\mu}{\alpha p_1} K_0(z) P_{0,k} \right], \)
Now, we express $P_{1,k}$, $P_{0,j}$ in terms of $P_{0,0}$. We observe that $z = 1$ and $z = 0$ are the roots of the denominator of the right hand side of the equations (25) and (28). We have $z = 1$ and $z = 0$ must be the roots of right hand side of those equations. Therefore

$$P_{1,k} = \xi_1 P_{0,0}, \text{ where } \xi_1 = \frac{(\xi - \alpha p_1)K_0(1)}{\mu_k K_1(1)}$$

(29)

And

$$P_{0,j} = C^j P_{0,0}, \text{ where } C = \frac{\phi K_1(1)}{(\xi - \alpha p_1)K_0(1)}, 1 \leq j \leq k - 1.$$  

(30)

Using (7) and (30), we obtain

$$P_{0,k} = \frac{\phi c^{k-1}}{\lambda} P_{0,0}.$$  

(31)

Using (29) and (30) in equations (25) and (28) respectively, we get

$$G_0(z) = \frac{1}{1 - \frac{\phi c^{k-1}}{\lambda} P_{0,0}} \left[ \frac{(\xi - \alpha p_1)K_0(z)}{\alpha p_1} - \frac{\mu}{\alpha p_1} K_1(z) \right] P_{0,0},$$

(32)

$$G_j(z) = \frac{1}{1 - \frac{\phi c^{k-1}}{\lambda} P_{0,0}} \left[ \frac{\xi - \alpha p_1}{\alpha p_1} K_0(z) - \frac{\phi}{C\alpha p_1} K_1(z) \right] P_{0,0} C^j, \quad 1 \leq j \leq k - 1.$$  

(33)

By taking $z = 1$ in equations (19) and (20), we obtain

$$- \beta G_k(1) + \phi \sum_{j=0}^{k-1} G_j(1) = - \beta P_{0,k} + \phi \sum_{j=0}^{k-1} P_{0,j} - \gamma G_{bd}(1)$$

(34)

And

$$G_{bd}(1) = \frac{\beta}{\gamma} \left[ G_k(1) - P_{0,k} \right].$$  

(35)

From equations (34) and (35), we have

$$\phi \sum_{j=0}^{k-1} G_j(1) = \mu P_{1,k} + \phi \sum_{j=0}^{k-1} P_{0,j} - 1.$$  

(36)

Using equations (29) and (30) in equation (36), we have

$$\sum_{j=0}^{k-1} G_j(1) = \left[ \frac{\mu c_1}{\phi} + \frac{1 - c^{k-1}}{1 - c} \right] P_{0,0}.$$  

(37)

Using equations (20) and (36) in equation (19), we obtain

$$G_k(z) = \frac{\phi \zeta \left[ 1 - G(z) \right] \sum_{j=0}^{k-1} [G_j(z) - G_j(1)] - N_i P_{0,k}}{\sum_{j=0}^{k-1} (\alpha p_1) \left[ (\xi - \alpha p_1)K_0(z) - \frac{\phi}{C\alpha p_1} K_1(z) \right] P_{0,0} C^j},$$

(38)

where

$$D_1(z) = \lambda z \left[ 1 - G(z) \right] \left[ \lambda(1 - G(z)) + \beta + \gamma \right] - \mu (1 - z) \left[ \lambda(1 - G(z)) + \gamma \right],$$

$$N_i = \left[ \mu (1 - z) - \beta z \right] \lambda(1 - G(z)) + \mu (1 - z) \gamma.$$  

Taking $z = 1$ and applying L-Hospital’s rule, we get

$$G_k(1) = \frac{\phi \sum_{j=0}^{k-1} c_j(1) + (\mu y - \beta \lambda G'(1)) P_{0,k}}{\mu y - \beta \lambda G'(1)}.$$  

(39)

From the normalization condition, we have that $\sum_{j=0}^{k-1} G_j(1) + G_k(1) + G_{bd}(1) = 1$. Using equations (35), (37) and (39), we have

$$P_{0,0} = \left[ M_1 + \left[ \frac{\phi \zeta}{\alpha p_1 + \phi} \left[ (L_1 + \frac{\mu c_1}{\phi}) + h(k) (CL_1 + L_2) \right] + M_2 \xi_0 \right] \left( \frac{1 + \phi}{M_3} \right) - \frac{\beta}{\gamma} \xi_0 \right]^{-1}$$

(40)

Where

$$M_1 = \frac{\mu c_1}{\phi} + \frac{1 - c^{k-1}}{1 - c}, \quad M_2 = \mu y - \beta \lambda G'(1), \quad M_3 = \mu y - \lambda(\beta + \gamma) G'(1),$$

$$L_1 = \xi - \alpha p_1, \quad L_2 = \frac{\lambda}{\alpha p_1} + \alpha p_1 - \xi.$$  

IV. PERFORMANCE MEASURES

In this section, we shall derive some performance measures as given below:

(i) Expected number of customers in the system during working vacation
\[ E[L_{WV}] = \sum_{j=0}^{k-1} G_j(1) = G_0'(1) + \sum_{j=1}^{k-1} G_j'(1) \]

\[ \sum_{j=1}^{k-1} G_j'(1) = \frac{h(k)}{\alpha p_1 + \phi} \left[ C(\xi - \alpha p_1) + \left(\frac{\lambda}{q} + \alpha p_1 - \xi\right)\right] P_{0,k} \]

where \( h(k) = \sum_{j=1}^{k} C^{j-1}. \)

(ii) Expected number of customers in the system during busy period

\[ E[L_k] = G_k'(1) = \frac{\phi_1 (\mu - \lambda (\beta + \gamma) C'(1))}{2(\lambda - \beta (\beta + \gamma))C'(1)} \sum_{j=0}^{k-1} G_j''(1) \left(1 + \frac{\phi_1 (\beta + \gamma) C''(1) + 2 \phi_1 (\mu - \beta (\beta + \gamma) C'(1))}{2(\lambda - \beta (\beta + \gamma) C'(1))^{2}}\right) \sum_{j=0}^{k-1} G_j'(1) \]

\[ + \frac{\lambda \mu^2 (2 \beta (\beta + \gamma) C'(1) + 2 \beta \lambda G'(1))}{2(\lambda - \beta (\beta + \gamma) C'(1))} P_{0,k} \]

where \( \sum_{j=0}^{k-1} G_j''(1) = -\frac{1}{2 \alpha p_1 + \phi}\left[\frac{2}{q} + \xi + \phi\right] P_{0,k} \sum_{j=0}^{k-1} G_j'(1) - \frac{2 \lambda}{q} \left(1 - \lambda C'(1)\right) \sum_{j=0}^{k-1} G_j(1). \)

(iii) The average number of customers during breakdown period is

\[ E[L_{bd}] = G_{bd}(1). \]

(iv) The average number of customers in the system is

\[ E[L] = E[L_{WV}] + E[L_k] + E[L_{bd}]. \]

(v) The rate of abundance of customers due to impatience

\[ R_a = \sum_{j=0}^{k-1} \sum_{n=2}^{\infty} (n-1) \alpha p_n P_{n,j}. \]

(vi) The probability of server being idle during regular busy period

\[ P_{0,k} = \frac{\phi_1}{\lambda} C^{k-1} P_{0,0}. \]

(vii) The probability that the server is on working vacation is

\[ P_{WV} = \sum_{j=0}^{k-1} G_j(1). \]

(viii) The probability that the server is busy and in breakdown state are given respectively, by

\[ P_b = \sum_{n=1}^{k} P_{n,k} = G_k(1) - P_{0,k} \quad \text{and} \quad P_{bd} = \sum_{n=1}^{\infty} P_{n,bd} = G_{bd}(1) \]

(ix) The average number of customers in the queue

\[ E[L_q] = \sum_{j=0}^{k} \sum_{n=1}^{\infty} (n-1) P_{n,j} + \sum_{n=1}^{\infty} n P_{n,bd} \]

\[ 4.1 \text{ Cost Model} \]

Now, we develop a total expected cost function per unit time with an objective to determine the optimum value of \( \mu \) that minimizes the expected cost function. Let us define:

\[ c_1 \equiv \text{Cost per unit time when the server is on working vacation}, \]

\[ c_2 \equiv \text{Cost per unit time when the server is busy}, \]

\[ c_3 \equiv \text{Cost per unit time when the server is on breakdown state}, \]

\[ c_4 \equiv \text{Cost per unit time when the server is on vacation completion}, \]

\[ c_5 \equiv \text{Cost per unit time when the server is serving in busy period}. \]
The cost minimization problem is expressed mathematically as:

\[ F[\mu] = c_1 * E[L_{WV}] + c_2 * E[L_k] + c_3 * E[L_{bd}] + c_4 * \phi + c_5 * \mu \]

4.2 Quadratic Fit Search Method

We employ the quadratic fit search method to solve the above optimization problem, as the computation of the above expected cost function derivatives is a non-trivial task.

Given a 3-point pattern, we can fit a quadratic function through corresponding functional values that has a unique minimum, \( x^q \), for the given objective function \( F[x] \). Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with approximate optimum \( x^q \). The unique optimum \( x^q \) of the quadratic function agreeing with \( F[x] \) at 3-point pattern \( (x^1, x^m, x^h) \) occurs at

\[
x^q \approx \frac{1}{2} \left[ \frac{F[x^1][s^m - s^h]}{[x^m - x^h]} + \frac{F[x^m][s^h - s^i]}{[x^h - x^i]} + \frac{F[x^h][s^i - s^m]}{[x^i - x^m]} \right]
\]

where

\[ s^i = (x^i)^2, s^m = (x^m)^2, s^h = (x^h)^2. \]

The steps of quadratic fit search method can be described as follows:

Step 1: Initialization. Choose a starting 3-point pattern \( (x^1, x^m, x^h) \) along with a stopping tolerance \( \varepsilon \) and initialize the iteration counter \( i = 1 \).

Step 2: Stopping. If \( |x^h - x^i| \leq \varepsilon \), stop and report approximate optimum solution \( x^m \).

Step 3: Quadratic fit. Compute a quadratic fit optimum \( x^q \) according to (41). Then if \( x^q < x^m \), go to Step 5, and if \( x^q > x^m \), go to Step 6.

Step 4: Coincide. New \( x^q \) coincides essentially with current \( x^m \). If \( x^q \) is farther from \( x^1 \) than from \( x^h \), perturb left

\[ x^q \leftarrow x^m - \frac{\varepsilon}{2} \]

And proceed to Step 5. Otherwise, adjust right

\[ x^q \leftarrow x^m + \frac{\varepsilon}{2} \]

And proceed to Step 6.

Step 5: Left. If \( F[x^m] \) is superior to \( F[x^q] \) (less for a minimization function, greater for a maximization function), then update \( x^i \leftarrow x^q \), otherwise replace \( x^h \leftarrow x^m, x^m \leftarrow x^q \).

Either way, advance \( i \leftarrow i + 1 \) and return to Step 2.

V. NUMERICAL INVESTIGATIONS

To demonstrate the applicability of the formulae obtained in the previous section, numerical computations have been carried out using Mathematica software. Some of the numerical computations have been presented below in the form of tables and graphs. The parameters of the model are assumed to be \( \lambda = 0.9, \mu = 2.5, \xi = 1.8, \alpha = 0.8, \phi = 2.0, \gamma = 5.0, \beta = 0.4, q = 0.6, k = 5 \) and \( p_1 = 0.7 \) unless their values are mentioned in the respective places.

We have taken the cost parameters as \( c_1 = 15, c_2 = 20, c_3 = 10, c_4 = 5, c_5 = 8 \) and tolerance limit for QFSM is \( \varepsilon = 1 \times 10^{-5} \).

Table 1 shows the effect of \( \mu \) on performance measures. As the service rate in busy period \( (\mu) \) increases, \( E[L_{WV}], R_{ad} \) and \( P_{0,k} \) increase. Since as \( \mu \) increases, customers served per unit time increases in busy period and server spends more time in vacation. Hence number of customers in working vacation \( E[L_{WV}] \) increases. As \( E[L_{WV}] \) increases, \( R_{ad} \) also increases since reneging occurs during working vacation. On the other hand, decreasing trend is observed for \( E[L_k], E[L_{bd}1] \) and \( E[L_d] \) which is intuitively true.

The effect of \( \xi \) on performance measures is shown in Table 2. As the service rate in working vacation \( (\xi) \) increases, \( E[L_{WV}], E[L_k], E[L_{bd1}], E[L_{bd}] \) and \( R_{ad} \) decrease. This is logical since higher service rate in working vacations results in lesser queue size in the system. On the other hand, \( P_{0,k} \) increases.
Table 3 shows the effect of $\Phi$ on performance measures. We can observe that as vacation rate ($\Phi$) increases, $E[L_k]$, $E[L_{bd}]$, $E[L_q]$ and $P_{0k}$ increase. This is due to the fact that as $\Phi$ increases, average vacation time decreases and server spends more duration of time in busy period. On the other hand, $E[L_{wy}]$ and $R_s$ decrease.

In Table 4, the effect of $\beta$ on performance measures is shown. As breakdown rate ($\beta$) increases, $E[L_{bd}]$, $E[L_k]$ and $E[L_q]$ increase and $E[L_{wy}]$, $R_s$ and $P_{0k}$ decrease. This is because, when system breakdowns it completely stops the service till it get repaired. Hence we can observe the corresponding increase and decrease in the performance measures.

The effect of retention probability ($q_1$) on $(\mu^*, E[\mu^*])$ is shown in Table 5 for different values of $\xi$. We can observe from the table that as $q_1$ increases, optimum value $\mu^*$ and $E[\mu^*]$ increase. This is because, in order to retain the potential customers in the queue, some trade-off with the increasing cost is necessary as observed in practical situations.
Table 5. Effect of retention probability \((q_s)\) on cost function

<table>
<thead>
<tr>
<th>(\xi)</th>
<th>(q_s = 0.2)</th>
<th>(q_s = 0.4)</th>
<th>(q_s = 0.6)</th>
<th>(q_s = 0.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.7</td>
<td>(3.944, 86.75)</td>
<td>(3.957, 87.26)</td>
<td>(3.970, 87.79)</td>
<td>(3.984, 88.35)</td>
</tr>
<tr>
<td>1.9</td>
<td>(3.928, 86.03)</td>
<td>(3.933, 86.49)</td>
<td>(3.946, 86.99)</td>
<td>(3.958, 87.51)</td>
</tr>
<tr>
<td>2.1</td>
<td>(3.902, 85.40)</td>
<td>(3.913, 85.84)</td>
<td>(3.925, 86.30)</td>
<td>(3.936, 86.78)</td>
</tr>
<tr>
<td>2.3</td>
<td>(3.885, 84.86)</td>
<td>(3.896, 85.27)</td>
<td>(3.906, 85.70)</td>
<td>(3.917, 86.15)</td>
</tr>
</tbody>
</table>

Figure 1 shows the effect of service rate in working vacation \((\xi)\) on \(E[L]\) for different values of reneging rate \((\alpha)\). It is clear that as \(\xi\) increases, \(E[L]\) decreases. Also as \(\alpha\) increases, the number of customers leaving the system increases. Hence we can see from the graph that \(E[L]\) decreases.

Figure 2 plots the effect of arrival rate \((\lambda)\) on probabilities of busy period and working vacation period \((P_b\text{ and } P_{WV})\). It is shown in the graph that as \(\lambda\) increases, \(P_b\) increases and \(P_{WV}\) decreases. This is due to the fact that increase in arrivals makes the system busy. Moreover, for fixed \(\mu\), the intersecting point of the curves of \(P_b\) and \(P_{WV}\) gives the value for which \(P_b\) is maximum and \(P_{WV}\) is minimum.
Figure 3. Effect of $\mu$ on $F[\mu]$

Figure 4. Effect of $\phi$ on $E[L_{nv}]$ for different $k$

Figure 5. Effect of $\phi$ on $E[L_{nv}]$ for different $k$
In Figure 3, the effect of $\phi$ on cost function is shown. It is found that the minimum value of the total expected cost function is at $\phi$. In Figures 4 and 5, the effect of vacation rate $\phi$ on $E[L_{WV}]$ and $E[L_k]$ is depicted for different values of $k$. It is clear that $E[L_{WV}]$ decreases and $E[L_k]$ increases with increase of $\phi$. It can be observed that the queue length during working vacation is lower in single working vacation ($k = 1$) than in multiple working vacations. This trend is reversed for the queue length during busy period.

VI. CONCLUSION

In this paper, we have carried out an analysis of MX/M/1 queue with variant working vacations, server breakdowns during busy period and retention of reneged customers during working vacations. Using probability generating function method, we obtained the number of customers in the system and corresponding mean system size during busy period. The effect of vacation rate on cost function is reversed for the queue length during busy period. In Figures 4 and 5, the effect of vacation rate is shown. It is found that the minimum value of the total expected cost function is at $\phi$. Finally, the effect of some parameters on performance measures of the system has been investigated and presented in the form of tables and graphs.

VII. REFERENCES