On the Construction of Vacuum Tubes

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Abstract
The implications of interactive information have been far-reaching and pervasive [1]. In fact, few physicists would disagree with the investigation of multi-processors [1]. Sope, our new system for replicated technology, is the solution to all of these obstacles.

Keywords: Vacuum Tubes, Local-Area Networks, Symmetric Encryption.

1. Introduction
In recent years, much research has been devoted to the visualization of Scheme; however, few have evaluated the extensive unification of hierarchical databases and IPv6. The notion that electrical engineers connect with pervasive symmetries is often considered essential. The notion that analysts connect with extreme programming is usually significant.

In this position paper, we describe new real-time technology (Sope), verifying that the much-touted efficient algorithm for the exploration of architecture by Kumar and Garcia [2] runs in $\Theta(\log n)$ time. Indeed, symmetric encryption and local-area networks have a long history of colluding in this manner. We view operating systems as following a cycle of four phases: synthesis, simulation, simulation, and study. In the opinion of computational biologists, the shortcoming of this type of solution, however, is that the well-known classical algorithm for the understanding of simulated annealing by Sasaki [2] runs in $\Theta(\log n)$ time. Combined with wearable configurations, such a claim constructs a perfect tool for evaluating the Ethernet.

End-users always deploy the location-identity split in the place of the investigation of sensor networks. Two properties make this approach distinct: Sope manages flexible algorithms, and also our heuristic constructs scatter/gather I/O. Sope creates Bayesian models. As a result, we see no reason not to use rasterization [3] to evaluate cooperative symmetries.

In this position paper, we make four main contributions. To begin with, we verify that the Ethernet can be made authenticated, replicated, and signed. We use real-time communication to prove that e-commerce and digital-to-analog converters can collude to fulfill this mission. We explore new cooperative archetypes (Sope), confirming that superblocks and Lamport clocks can cooperate to overcome this riddle. In the end, we demonstrate that though active networks [4] can be made efficient, multimodal, and encrypted, the infamous ubiquitous algorithm for the analysis of object-oriented languages by Manuel Blum et al. is NP-complete.

The rest of the paper proceeds as follows. Primarily, we motivate the need for congestion control. Next, we argue the emulation of IPv6. Continuing with this rationale, we demonstrate the important unification of multicast applications and object-oriented languages. Finally, we conclude.

2. Framework
Suppose that there exists 16 bit architectures [5] such that we can easily analyze scatter/gather I/O [6,7,8]. Consider the early framework by Martin et al.; our methodology is similar, but will actually achieve this mission. Sope does not require such a typical synthesis to run correctly, but it doesn't hurt. This seems to hold in most cases. Next, rather than learning consistent hashing, Sope chooses to locate flexible methodologies. This is a confirmed property of our framework. Thus, the methodology that our system uses is unfounded.
Despite the results by Richard Hamming, we can disprove that massive multiplayer online role-playing games and 32 bit architectures are regularly incompatible. Consider the early methodology by Williams; our model is similar, but will actually fulfill this aim. See our prior technical report \[9\] for details.

Theorem 1.1: If \( K \) is a compact set in the plane whose complement is connected, if \( f \) is a continuous complex function on \( K \) which is holomorphic in the interior of \( K \), and if \( 0, \epsilon > 0 \), then there exists a polynomial \( P \) such that

\[
|f(z) - P(z)| < \epsilon K \quad (z \in K) \tag{2}
\]

By (1), this proves the theorem. Our first objective is the construction of a function \( \Phi \in \mathcal{C}_c(R^2) \), such that for all \( z \)

\[
|f(z) - \Phi(z)| \leq \epsilon K, \quad (3)
\]

\[
|\partial \Phi(z)| < \frac{2\epsilon K}{\delta}, \quad (4)
\]

And

\[
\Phi(z) = \frac{1}{\pi} \int_{\delta}^{2\pi} (\partial \Phi)(\xi) \, d\xi d\eta \quad (\xi = \xi + i\eta), \tag{5}
\]

Where \( X \) is the set of all points in the support of \( \Phi \) whose distance from the complement of \( K \) does not \( \delta \). (Thus \( X \) contains no point which is “far within” \( K \).) We construct \( \Phi \) as the convolution of \( f \) with a smoothing function \( A \). Put \( a(r) = 0 \) if \( r > \delta \), put

\[
a(r) = \frac{3}{\pi \delta^2} (1 - \frac{r^2}{\delta^2})^2 \quad (0 \leq r \leq \delta), \quad (6)
\]

And define

\[
A(z) = a(|z|) \tag{7}
\]

For all complex \( z \). It is clear that \( A \in \mathcal{C}_c(R^2) \). We claim that

\[
\int_{K'} A = 1, \quad (8)
\]

\[
\int_{K'} \partial A = 0, \quad (9)
\]

\[
\int_{K'} |\partial A| = \frac{24}{15\delta} < \frac{2}{\delta} \quad (10)
\]

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates), (9) holds simply because \( A \) has compact support. To compute (10), express \( \partial A \) in polar coordinates, and note that

\[
\frac{\partial A}{\partial \theta} = 0,
\]

\[
\frac{\partial A}{\partial r} = -a',
\]

Now define
\[ \Phi(z) = \int_{K} f(z - \zeta) A d\zeta d\eta = \int_{K} [f(z - \zeta) - f(z)] A(\zeta) d\zeta d\eta \tag{11} \]

Since \( f \) and \( A \) have compact support, so does \( \Phi \).

Since \( \Phi(z) - f(z) = \int_{K} [f(z - \zeta) - f(z)] A(\zeta) d\zeta d\eta \tag{12} \)

And \( A(\zeta) = 0 \) if \( |\zeta| > \delta \), (3) follows from (8).

The difference quotients of \( A \) converge boundedly to the corresponding partial derivatives[1-50], since \( A \in C_{0}^{\infty}(R^{2}) \). Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

\[ (\partial \Phi)(z) = \int_{K} (\partial A)(z - \zeta)f(\zeta) d\zeta d\eta \]
\[ = \int_{K} f(z - \zeta)(\partial A)(\zeta) d\zeta d\eta \]
\[ = \int_{K} [f(z - \zeta) - f(z)] (\partial A)(\zeta) d\zeta d\eta \tag{13} \]

The last equality depends on (9). Now (10) and (13) give (4).

If we write (13) with \( \Phi_{x} \) and \( \Phi_{y} \) in place of \( \Phi \), we see that \( \Phi \) has continuous partial derivatives[1-161], since \( A \in C_{0}^{\infty}(R^{2}) \). Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

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\[ = \int_{K} [f(z - \zeta) - f(z)] (\partial A)(\zeta) d\zeta d\eta \tag{13} \]
\[ F(z) - \Phi(z) < \frac{\alpha(\delta)}{\pi \delta} \int_\chi |R(\zeta, z)| \]

\[-\frac{1}{z - \zeta} |d\xi d\eta| (z \in \Omega) \quad (22)\]

Observe that the inequalities (16) and (17) are valid with \( R \) in place of \( Q \) if \( \zeta \in X \) and \( z \in \Omega \). Now fix \( \zeta \in \Omega \), put \( \zeta = z + pe^{i\theta} \), and estimate the integrand in (22) by (16) if \( 4\delta < \rho \) by (17) if \( 4\delta \leq \rho \). The integral in (22) is then seen to be less than the sum of

\[ \int_0^{4\delta} \frac{50}{\delta} \rho d\rho = 808\pi\delta \quad (23) \]

And

\[ \int_0^{4\delta} \frac{4000\delta^2}{\rho^2} \rho d\rho = 2000\pi\delta. \quad (24) \]

Hence (22) yields

\[ |F(z) - \Phi(z)| < 6000\alpha(\delta) \quad (z \in \Omega) \quad (25) \]

Since \( F \in H(\Omega) \), \( K \subset \Omega \), and \( S = K \) is connected, Runge’s theorem shows that \( F \) can be uniformly approximated on \( K \) by polynomials. Hence (3) and (25) show that (2) can be satisfied. This completes the proof.

Lemma 1.0: Suppose \( f \in C_c^\infty (\mathbb{R}^2) \), the space of all continuously differentiable functions in the plane, with compact support. Put

\[ \partial = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad (1) \]

Then the following “Cauchy formula” holds:

\[ f(z) = -\frac{1}{\pi} \int \frac{(\partial f)(\zeta)}{\zeta - z} d\xi d\eta \quad (\zeta = \xi + i\eta) \quad (2) \]

Proof: This may be deduced from Green’s theorem. However, here is a simple direct proof:

Put \( \varphi(r, \theta) = f(z + re^{i\theta}) \), \( r > 0 \), \( \theta \) real

If \( \zeta = z + re^{i\theta} \), the chain rule gives

\[ (\partial f)(\zeta) = \frac{1}{2} e^{i\theta} \left[ \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right] \varphi(r, \theta) \quad (3) \]

The right side of (2) is therefore equal to the limit, as \( \varepsilon \to 0 \), of

\[ -\frac{1}{\pi} \int_0^{2\pi} \left( \frac{\partial \varphi}{\partial r} + \frac{i}{r} \frac{\partial \varphi}{\partial \theta} \right) d\theta dr \quad (4) \]

For each \( r > 0 \), \( \varphi \) is periodic in \( \theta \), with period \( 2\pi \). The integral of \( \partial \varphi / \partial \theta \) is therefore 0, and (4) becomes

\[ -\frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^{2\pi} \varphi(e, \theta) e^{i\theta} d\theta \quad (5) \]

As \( \varepsilon \to 0 \), \( \varphi(e, \theta) \to f(z) \) uniformly. This gives (2)

If \( X^\alpha \in A \) and \( X^\beta \in k[X, \ldots, X_n] \), then

\[ X^\alpha X^\beta = X^\alpha + \beta = X^a \quad (\text{ideal}) \]

and so \( A \) satisfies the condition (*). Conversely, \( (\sum_{a\in A} X^a) (\sum_{b\in A} X^b) = \sum_{a+b\in A} X^{a+b} \) (finite sums), and so if \( A \) satisfies (*), then the subspace generated by the monomials \([1, 161] X^a, \alpha \in A \), is an ideal. The proposition gives a classification of the monomial ideals in \( k[X, \ldots, X_n] \): they are in one to one correspondence with the subsets \( A \) of \( \square_n \) satisfying (*). For example, the monomial ideals in \( k[X] \) are exactly the ideals \( (X^n), n \geq 1 \), and the zero ideal (corresponding to the empty set \( A \)). We write \( \langle X^\alpha | \alpha \in A \rangle \) for the ideal corresponding to \( A \) (subspace generated by the \( X^\alpha, \alpha \in A \)).

**Lemma 1.1.** Let \( S \) be a subset of \( \square_n \). The ideal \( A \) generated by \( X^a, \alpha \in S \) is the monomial ideal corresponding to...
A\{ \beta \in \mathbb{N}^n \mid \beta - \alpha \in \mathbb{N}^n, \text{ some } \alpha \in S \}

Thus, a monomial is in \( A \) if and only if it is divisible by one of the \( X^\alpha, \alpha \in S \).

**PROOF.** Clearly \( A \) satisfies \((*)\), and \( a \subset \{ X^\beta \mid \beta \in A \} \). Conversely, if \( \beta \in A \), then \( \beta - \alpha \in \mathbb{N}^n \) for some \( \alpha \in S \), and \( X^\beta = X^\alpha X^{\beta - \alpha} \in a \). The last statement follows from the fact that \( X^\alpha \mid X^\beta \iff \beta - \alpha \in \mathbb{N}^n \). Let \( A \subset \mathbb{N}^n \) satisfy \((*)\). From the geometry of \( A \), it is clear that there is a finite set of elements \( \{ \alpha_1, \ldots, \alpha_s \} \) of \( A \) such that \( A = \{ \beta \in \mathbb{N}^n \mid \beta - \alpha_i \in \mathbb{Z}^2, \text{ some } \alpha_i \in S \} \).

(The \( \alpha_i \)'s are the corners of \( A \)). Moreover, \( a = \{ X^\alpha \mid \alpha \in A \} \) is generated by the monomials \( X^\alpha, \alpha \in S \).

**DEFINITION 1.0.** For a nonzero ideal \( a \) in \( k[X_1, \ldots, X_n] \), we let \((LT(a))\) be the ideal generated by \( \{LT(f) \mid f \in a \} \).

**LEMMA 1.2** Let \( a \) be a nonzero ideal in \( k[X_1, \ldots, X_n] \), then \((LT(a))\) is a monomial ideal, and it equals \( (LT(g_1), \ldots, LT(g_s)) \) for some \( g_1, \ldots, g_s \in a \).

**PROOF.** Since \((LT(a))\) can also be described as the ideal generated by the leading monomials (rather than the leading terms) of elements of \( a \).

**THEOREM 1.2.** Every ideal \( a \) in \( k[X_1, \ldots, X_n] \) is finitely generated; more precisely, \( a = \langle g_1, \ldots, g_s \rangle \) where \( g_1, \ldots, g_s \) are any elements of \( a \) whose leading terms generate \( LT(a) \).

**PROOF.** Let \( f \in a \). On applying the division algorithm, we find \( f = a_i g_i + \ldots + a_s g_s + r \), \( a_i, r \in k[X_1, \ldots, X_n] \), where either \( r = 0 \) or no monomial occurring in it is divisible by any \( LT(g_i) \). But \( r = f - \sum a_i g_i \in a \), and therefore \( LT(r) \in LT(a) = (LT(g_1), \ldots, LT(g_s)) \), implies that every monomial occurring in \( r \) is divisible by one in \( LT(g_i) \). Thus \( r = 0 \), and \( g \in \langle g_1, \ldots, g_s \rangle \).

**DEFINITION 1.1.** A finite subset \( S = \{ g_1, \ldots, g_s \} \) of an ideal \( a \) is a standard (Gröbner) basis for \( a \) if \( (LT(g_1), \ldots, LT(g_s)) = LT(a) \). In other words, \( S \) is a standard basis if the leading term of every element of \( a \) is divisible by at least one of the leading terms of the \( g_i \).

**THEOREM 1.3** The ring \( k[X_1, \ldots, X_n] \) is Noetherian i.e., every ideal is finitely generated.

**PROOF.** For \( n = 1 \), \( k[X] \) is a principal ideal domain, which means that every ideal is generated by single element. We shall prove the theorem by induction on \( n \). Note that the obvious map \( k[X_1, \ldots, X_{n-1}] \to k[X_1, \ldots, X_n] \) is an isomorphism – this simply says that every polynomial \( f \) in \( n \) variables \( X_1, \ldots, X_n \) can be expressed uniquely as a polynomial in \( X_{n-1} \) with coefficients in \( k[X_1, \ldots, X_{n-1}] \):

\[ f(X_1, \ldots, X_n) = a_0(X_1, \ldots, X_{n-1})X_n + \ldots + a_{n-1}(X_1, \ldots, X_{n-1}) \]

Thus the next lemma will complete the proof.

**LEMMA 1.3.** If \( A \) is Noetherian, then so also is \( A[X] \).

**PROOF.** For a polynomial
Let $a$ be an ideal in $A[X]$. The leading coefficients of the polynomials in $a$ form an ideal $a'$ in $A$, and since $A$ is Noetherian, $a'$ will be finitely generated. Let $g_1, \ldots, g_m$ be elements of $a'$ whose leading coefficients generate $a'$, and let $r$ be the maximum degree of $g_i$. Now let $f \in a$, and suppose $f$ has degree $s > r$, say, $f = aX^s + \ldots$. Then $a \in a'$, and so we can write 

$$a = \sum b_i g_i,$$

where $b_i \in A$, $a_i =$leading coefficient of $g_i$.

Now 

$$f - \sum b_i g_i X^{r-i}, \quad r = \deg(g_i),$$

has degree $< \deg(f)$. By continuing in this way, we find that $f \equiv f_i \mod(g_1, \ldots, g_m)$ With $f_i$, a polynomial of degree $t < r$. For each $d < r$, let $a_d$ be the subset of $A$ consisting of 0 and the leading coefficients of all polynomials in $a$ of degree $d$; it is again an ideal in $A$. Let $g_{d,1}, \ldots, g_{d,m_d}$ be polynomials[1-161] of degree $d$ whose leading coefficients generate $a_d$. Then the same argument as above shows that any polynomial $f_d$ in $a$ of degree $d$ can be written 

$$f_d \equiv f_{d-1} \mod(g_{d,1}, \ldots, g_{d,m_d})$$

With $f_{d-1}$ of degree $\leq d - 1$. On applying this remark repeatedly we find that 

$$f_i \in (g_{r-1,1}, \ldots, g_{r-1,m_{r-1}}, \ldots, g_{0,1}, \ldots, g_{0,m_0})$$

Hence 

$$f_i \in (g_1, \ldots, g_m, g_{r-1,1}, \ldots, g_{r-1,m_{r-1}}, \ldots, g_{0,1}, \ldots, g_{0,m_0})$$

and so the polynomials $g_1, \ldots, g_{m_0}$ generate $a$.

One of the great successes of category theory in computer science has been the development of a “unified theory” of the constructions underlying denotational semantics. In the untyped $\lambda$-calculus, any term may appear in the function position of an application. This means that a model $D$ of the $\lambda$-calculus[1-161] must have the property that given a term $t$ whose interpretation is $d \in D$. Also, the interpretation of a functional abstraction like $\lambda x.X$ is most conveniently defined as a function from $D$ to $D$, which must then be regarded as an element of $D$. Let $\psi : [D \to D] \to D$ be the function that picks out elements of $D$ to represent elements of $[D \to D]$ and $\phi : D \to [D \to D]$ be the function that maps elements of $D$ to functions of $D$. Since $\psi(f)$ is intended to represent the function $f$ as an element of $D$, it makes sense to require that $\phi(\psi(f)) = f$, that is, $\psi \circ \phi = id_{[D \to D]}$.

Furthermore, we often want to view every element of $D$ as representing some function from $D$ to $D$ and require that elements representing the same function be equal – that is 

$$\psi(\phi(d)) = d$$

or 

$$\psi \circ \phi = id_D$$

The latter condition is called extensionality. These conditions together imply that $\phi$ and $\psi$ are inverses --- that is, $D$ is isomorphic to the space of functions from $D$ to $D$ that can be the interpretations of functional abstractions: $D \cong [D \to D]$. Let us suppose we are working with the untyped $\lambda$-calculus. we need a solution of the equation 

$$D \cong A + [D \to D],$$

where $A$ is some predetermined domain containing interpretations for elements of $C$. Each element of $D$ corresponds to either an element of $A$ or an element of $[D \to D]$, with a tag. This equation can be solved by finding least fixed points of the function $F(X) = A + [X \to X]$ from domains to domains --- that is, finding domains $X$ such that $X \cong A + [X \to X]$, and such that for any domain

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Y also satisfying this equation, there is an embedding of \( X \) to \( Y \) --- a pair of maps
\[
\begin{align*}
Y \xrightarrow{f} X & \xrightarrow{f^*} Y \\
Y \xrightarrow{f^*} X & \xrightarrow{f^*} Y
\end{align*}
\]
Such that
\[
\begin{align*}
f^* \circ f & = id_X \\
f \circ f^* & \subseteq id_Y
\end{align*}
\]
Where \( f \subseteq g \) means that \( f \) approximates \( g \) in some ordering representing their information content. The key shift of perspective from the domain-theoretic[1-161] to the more general category-theoretic approach lies in considering \( F \) not as a function on domains, but as a functor on a category of domains. Instead of a least fixed point of the function, \( F \).

**Definition 1.3**: Let \( K \) be a category and \( F : K \to K \) as a functor. A fixed point of \( F \) is a pair \((A,a)\), where \( A \) is a \( K \)-object and \( a : F(A) \to A \) is an isomorphism. A prefixed point of \( F \) is a pair \((A,a)\), where \( A \) is a \( K \)-object and \( a \) is any arrow from \( F(A) \) to \( A \).

**Definition 1.4**: An \( \omega \)-chain in a category \( K \) is a diagram of the following form:
\[
\Delta = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} D_2 \xrightarrow{f_2} \ldots \]
Recall that a cocone \( \mu \) of an \( \omega \)-chain \( \Delta \) is a \( K \)-object \( X \) and a collection of \( K \)-arrows \( \{ \mu_i : D_i \to X \mid i \geq 0 \} \) such that for all \( i \geq 0 \), \( \mu_i = f_{i+1} \circ \mu_{i+1} \). An \( \omega \)-chain \( \Delta \) is also a cone, then there exists a unique mediating arrow \( k : X \to X \) such that for all \( i \geq 0 \), \( \mu_i = k \circ \mu_{i+1} \). We write \( \downarrow \) (or just \( \downarrow \) ) for the distinguish initial object of \( K \), when it has one, and \( \downarrow \to A \) for the unique arrow from \( \downarrow \) to each \( K \)-object \( A \). It is also convenient to write
\[
\Delta^\perp = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \ldots \to 
\]
...to denote all of \( \Delta \) except \( D_0 \) and \( f_0 \). By analogy, \( \mu^\perp \) is \( \{ \mu_i \mid i \geq 1 \} \).

We write \( F^i \) for the \( i \)-fold iterated composition of \( F \) --- that is,
\[
F^0(f) = f, F^1(f) = F(f), F^2(f) = F(F(f)),
\]
...etc. With these definitions we can state that every non-iteric function on a complete lattice has a least fixed point:

**Lemma 1.4**: Let \( K \) be a category with initial object \( \perp \) and let \( F : K \to K \) be a functor. Define the \( \omega \)-chain \( \Delta \) by
\[
\Delta = \perp \to F(\perp) \xrightarrow{F(\perp)} F^2(\perp) \xrightarrow{F^2(\perp)} \ldots
\]
If both \( \mu : \Delta \to D \) and \( F(\mu) : F(\Delta) \to F(D) \) are colimits, then \((D,d)\) is an initial \( F \)-algebra, where \( d : F(D) \to D \) is the mediating arrow from
\( F(\mu) \) to the cocone \( \mu^\perp \).

**Theorem 1.4**: Let a DAG \( G \) given in which each node is a random variable, and let a discrete conditional probability distribution of each node given values of its parents in \( G \) be specified. Then the

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product of these conditional distributions yields a joint probability distribution \( P \) of the variables, and \((G,P)\) satisfies the Markov condition.

**Proof.** Order the nodes according to an ancestral ordering. Let \( X_1, X_2, \ldots, X_n \) be the resultant ordering. Next define:

\[
P(x_1, x_2, \ldots, x_n) = P(x_1 \mid pa_{x_1})P(x_2 \mid pa_{x_2}) \ldots P(x_n \mid pa_{x_n}).
\]

Where \( PA_i \) is the set of parents of \( X_i \) of in \( G \) and \( P(x_i \mid pa_{x_i}) \) is the specified conditional probability distribution. First we show this does indeed yield a joint probability distribution. Clearly, \( 0 \leq P(x_1, x_2, \ldots, x_n) \leq 1 \) for all values of the variables. Therefore, to show we have a joint distribution, as the variables range through all their possible values, is equal to one. To that end, Specified conditional distributions are the conditional distributions they notationally represent in the joint distribution. Finally, we show the Markov condition is satisfied. To do this, we need show for \( 1 \leq k \leq n \) that whenever

\[
P(pa_{x_k}) \neq 0 \text{ if } P(nd_{x_k} \mid pa_{x_k}) \neq 0
\]

and

\[
P(x_k \mid pa_{x_k}) \neq 0
\]

then

\[
P(x_1 \mid nd_{x_k}, pa_{x_k}) = P(x_k \mid pa_{x_k}),
\]

Where \( ND_k \) is the set of nondescendents of \( X_k \) of in \( G \). Since \( PA_k \subseteq ND_k \), we need only show

\[
P(x_k \mid nd_{x_k}, pa_{x_k}) = P(x_k \mid pa_{x_k}).
\]

First for a given \( k \), order the nodes so that all and only nondescendents of \( X_k \) precede \( X_k \) in the ordering. Note that this ordering depends on \( k \), whereas the ordering in the first part of the proof does not. Clearly then

\[
ND_k = \{X_1, X_2, \ldots, X_{k-1}\}
\]

Let

\[
D_k = \{X_{k+1}, X_{k+2}, \ldots, X_n\}
\]

follows

\[
\sum_{x_k} P(x_1, x_2, \ldots, x_n)
\]

We define the \( m^{th} \) cyclotomic field to be the field \( \mathbb{Q}[x] / (\Phi_m(x)) \) where \( \Phi_m(x) \) is the \( m^{th} \) cyclotomic polynomial. \( \mathbb{Q}[x] / (\Phi_m(x)) \) has degree \( \varphi(m) \) over \( \mathbb{Q} \), since \( \Phi_m(x) \) has degree \( \varphi(m) \). The roots of \( \Phi_m(x) \) are just the primitive \( m^{th} \) roots of unity, so the complex embeddings of \( \mathbb{Q}[x] / (\Phi_m(x)) \) are simply the \( \varphi(m) \) maps

\[
\sigma_k : \mathbb{Q}[x] / (\Phi_m(x)) \mapsto C,
\]

\( 1 \leq k \leq m, (k, m) = 1 \), where

\[
\sigma_k(x) = \xi_m^k,
\]

\( \xi_m \) being our fixed choice of primitive \( m^{th} \) root of unity. Note that \( \xi_m \in \mathbb{Q}(\xi_m) \) for every \( k \); it follows that

\[
\mathbb{Q}(\xi_m) = \mathbb{Q}(\xi_m^k) \text{ for all } k \text{ relatively prime to } m.
\]

In particular, the images of the \( \sigma \), coincide, so \( \mathbb{Q}[x] / (\Phi_m(x)) \) is Galois over \( \mathbb{Q} \). This means that we can write \( \mathbb{Q}(\xi_m) \) for \( \mathbb{Q}[x] / (\Phi_m(x)) \) without much fear of ambiguity; we will do so from now on, the identification being \( \xi_m \mapsto x \). One advantage of this is that one can easily talk about cyclotomic fields being extensions of one another, or intersections or compositums; all of these things take place considering them as subfield of \( C \). We now investigate some basic properties of cyclotomic [1-161] fields. The first issue is whether or not they are all distinct; to determine this, we need to know which roots of unity lie in \( \mathbb{Q}(\xi_m) \). Note, for example, that if \( m \) is odd, then \( -\xi_m \) is a \( 2m^{th} \) root of unity. We will show that this is the only way in which one can obtain any non-\( m^{th} \) roots of unity.

**Lemma 1.5** If \( m \) divides \( n \), then \( \mathbb{Q}(\xi_m) \) is contained in \( \mathbb{Q}(\xi_n) \)

**Proof.** Since \( \xi_n^{m/n} = \xi_m \), we have \( \xi_m \in \mathbb{Q}(\xi_n) \), so the result is clear.
LEMMA 1.6 If \( m \) and \( n \) are relatively prime, then
\[
Q(\xi_m, \xi_n) = Q(\xi_{mn})
\]
and
\[
Q(\xi_m) \cap Q(\xi_n) = Q
\]
(Recall the \( Q(\xi_m, \xi_n) \) is the compositum of \( Q(\xi_m) \) and \( Q(\xi_n) \)).

PROOF. One checks easily that \( \xi_m \xi_n \) is a primitive \( mn \)th root of unity, so that
\[
Q(\xi_m, \xi_n) \subseteq Q(\xi_{mn})
\]
and
\[
[Q(\xi_m, \xi_n) : Q] \leq [Q(\xi_{mn}) : Q][Q(\xi_m : Q) = \phi(m)\phi(n) = \phi(mn);
\]
Since
\[
[Q(\xi_{mn}) : Q] = \phi(mn);
\]
this implies that
\[
Q(\xi_m, \xi_n) = Q(\xi_{mn}) \quad \text{We know that } Q(\xi_m, \xi_n) \quad \text{has degree } \phi(mn) \quad \text{over } Q, \quad \text{so we must have}
\]
\[
[Q(\xi_m, \xi_n) : Q(\xi_{mn})] = \phi(n)
\]
and
\[
[Q(\xi_m, \xi_n) : Q(\xi_{mn})] = \phi(m)
\]
\[
[Q(\xi_{mn}) : Q(\xi_{mn}) \cap Q(\xi_n)] \geq \phi(m);
\]
And thus that
\[
Q(\xi_{mn}) \cap Q(\xi_n) = Q
\]
PROPOSITION 1.2 For any \( m \) and \( n \)
\[
Q(\xi_m, \xi_n) = Q(\xi_{[m,n]})
\]
And
\[
Q(\xi_m) \cap Q(\xi_n) = Q(\xi_{[m,n]})
\]
here \([m,n]\) and \((m,n)\) denote the least common multiple and the greatest common divisor of \( m \) and \( n \), respectively.

PROOF. Write \( m = p_1^{e_1} \cdots p_k^{e_k} \) and \( n = p_1^{f_1} \cdots p_k^{f_k} \) where the \( p_i \) are distinct primes. (We allow \( e_i \) or \( f_i \) to be zero)
\[
Q(\xi_m) = Q(\xi_{p_1^{e_1}})Q(\xi_{p_2^{e_2}}) \cdots Q(\xi_{p_k^{e_k}})
\]
and
\[
Q(\xi_n) = Q(\xi_{p_1^{f_1}})Q(\xi_{p_2^{f_2}}) \cdots Q(\xi_{p_k^{f_k}})
\]
Thus
\[
Q(\xi_m, \xi_n) = Q(\xi_{p_1^{\min(e,f)}})Q(\xi_{p_2^{\min(e,f)}}) \cdots Q(\xi_{p_k^{\min(e,f)}})
\]
An entirely similar computation shows that 
\[
Q(\xi_{mn}) \cap Q(\xi_n) = Q(\xi_{[m,n]})
\]
Mutual information measures the information transferred when \( x_i \) is sent and \( y_i \) is received, and is defined as
\[
I(x_i, y_i) = \log_2 \frac{P(x_i)}{P(y_i)} \quad \text{bits} \quad (1)
\]
In a noise-free channel, each \( y \) is uniquely connected to the corresponding \( x \), and so they constitute an input–output pair \((x_i, y_i)\) for which
\[
P \left( \frac{x_i}{y_i} \right) = 1 \quad \text{and} \quad I(x_i, y_i) = \log_2 \frac{1}{P(x_i)} \quad \text{bits}
\]
that is, the transferred information is equal to the self-information that corresponds to the input \( x \). In a very noisy channel, the output \( y \) and input \( x \) would be completely uncorrelated, and so \( P \left( \frac{x_i}{y_i} \right) = P(x_i) \) and also \( I(x_i, y_i) = 0 \); that is, there is no transference of information. In general, a given channel will operate between these two extremes. The mutual information is defined between the input

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and the output of a given channel. An average of the calculation of the mutual information for all input-output pairs of a given channel is the average mutual information:

$$I(X,Y) = \sum_{i,j} P(x_i,y_j) I(x_i,y_j) = \sum_{i,j} P(x_i,y_j) \log_2 \left[ \frac{P(x_i)/y_j}{P(x_i)} \right]$$

bits per symbol. This calculation is done over the input and output alphabets. The average mutual information. The following expressions are useful for modifying the mutual information expression:

$$P(x_i,y_j) = P\left(\frac{x_i}{y_j}\right)P(y_j) = P\left(\frac{y_j}{x_i}\right)P(x_i)$$

$$P(y_j) = \sum_j P\left(\frac{y_j}{x_i}\right)P(x_i)$$

$$P(x_i) = \sum_i P\left(\frac{x_i}{y_j}\right)P(y_j)$$

Then

$$I(X,Y) = \sum_{i,j} P(x_i,y_j)$$

$$= \sum_{i,j} P(x_i,y_j) \log_2 \left[ \frac{1}{P(x_i)} \right]$$

$$-\sum_{i,j} P(x_i,y_j) \log_2 \left[ \frac{1}{P\left(\frac{x_i}{y_j}\right)} \right]$$

$$\sum_{i,j} \left[ P\left(\frac{x_i}{y_j}\right)P(y_j) \right] \log_2 \frac{1}{P(x_i)}$$

$$\sum_i P(x_i) \log_2 \frac{1}{P(x_i)} = H(X)$$

$$I(X,Y) = H(X) - H\left(\frac{X}{Y}\right)$$

where

$$H\left(\frac{X}{Y}\right) = \sum_{i,j} P(x_i,y_j) \log_2 \left[ \frac{1}{P\left(\frac{x_i}{y_j}\right)} \right]$$

is usually called the equivocation. In a sense, the equivocation can be seen as the information lost in the noisy channel, and is a function of the backward conditional probability. The observation of an output symbol $y_j$ provides $H(X) - H\left(\frac{X}{Y}\right)$ bits of information. This difference is the mutual information of the channel. Mutual Information: Properties Since

$$P\left(\frac{x_i}{y_j}\right)P(y_j) = P\left(\frac{y_j}{x_i}\right)P(x_i)$$

The mutual information fits the condition

$$I(X,Y) = I(Y,X)$$

And by interchanging input and output it is also true that

$$I(X,Y) = H(Y) - H\left(\frac{Y}{X}\right)$$

Where

$$H(Y) = \sum_j P(y_j) \log_2 \frac{1}{P(y_j)}$$

This last entropy is usually called the noise entropy. Thus, the information transferred through the channel is the difference between the output entropy and the noise entropy. Alternatively, it can be said that the channel mutual information\[1-161\] is the difference between the number of bits needed for determining a given input symbol before knowing the corresponding output symbol, and the number of bits needed for determining a given input symbol after knowing the corresponding output symbol

$$I(X,Y) = H(X) - H\left(\frac{X}{Y}\right)$$

As the channel mutual information expression is a difference between two quantities, it seems that this parameter can adopt negative values. However, and is spite of the fact that for some $y_j$, $H(X / y_j)$ can be larger than $H(X)$, this is not possible for the average value calculated over all the outputs:

$$\sum_{i,j} P(x_i,y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} = \sum_{i,j} P(x_i,y_j) \log_2 \frac{P(x_i,y_j)}{P(x_i)P(y_j)}$$

Then

$$-I(X,Y) = \sum_{i,j} P(x_i,y_j) \frac{P(x_i)P(y_j)}{P(x_i,y_j)} \leq 0$$

Because this expression is of the form
The above expression can be applied due to the factor $P(x_i)P(y_j)$, which is the product of two probabilities, so that it behaves as the quantity $Q_i$, which in this expression is a dummy variable that fits the condition $\sum_i Q_i \leq 1$. It can be concluded that the average mutual information is a non-negative number. It can also be equal to zero, when the input and the output are independent of each other. A related entropy called the joint entropy is defined as

$$H(X,Y) = \sum_{i,j} P(x_i,y_j) \log_2 \frac{1}{P(x_i,y_j)}$$

$$= \sum_{i} P(x_i) \sum_{j} P(y_j | x_i) \log_2 \frac{1}{P(y_j | x_i)}$$

Theorem 1.5: Entropies of the binary erasure channel (BEC) The BEC is defined with an alphabet of two inputs and three outputs, with symbol probabilities.

$P(x_1) = \alpha$ and $P(x_2) = 1 - \alpha$, and transition probabilities

$P(y_1/x_1) = 1 - p$ and $P(y_1/x_2) = 0$,
and $P(y_2/x_1) = 0$
and $P(y_2/x_2) = p$
and $P(y_3/x_2) = 1 - p$

Lemma 1.7. Given an arbitrary restricted time-discrete, amplitude-continuous channel whose restrictions are determined by sets $F_n$ and whose density functions exhibit no dependence on the state space, let $n$ be a fixed positive integer, and $p(x)$ an arbitrary probability density function on Euclidean $n$-space. $p(x | y)$ for the density $p_n(y_1,...,y_n | x_1,...,x_n)$ and $F$ for $F_n$. For any real number $a$, let

$$A = \left\{ (x, y) : \log \frac{p(y | x)}{p(y)} > a \right\} \quad (1)$$

Then for each positive integer $u$, there is a code $(u, n, \lambda)$ such that

$$\lambda \leq u e^{-u} + P\{ (X, Y) \notin A \} + P\{ X \notin F \} \quad (2)$$

where

$$P\{ (X, Y) \in A \} = \int_{x,y} p(x,y) dx dy, \quad p(x,y) = p(x)p(y | x)$$

and

$$P\{ X \notin F \} = \int_{x} p(x) dx$$

Proof: A sequence $x^{(1)} \in F$ such that

$$P\{ Y \in A_x \mid X = x^{(1)} \} \geq 1 - \epsilon$$

where $A_x = \{ y : (x,y) \notin A \}$.

Choose the decoding set $B_1$ to be $A_{x^{(1)}}$. Having chosen $x^{(1)},...,x^{(k-1)}$ and $B_1,...,B_{k-1}$, select $x^k \in F$ such that

$$P\{ Y \in A_{x^k} \mid -\bigcup_{i=1}^{k-1} B_i \mid X = x^{(k)} \} \geq 1 - \epsilon;$$

Set $B_k = A_{x^k} \setminus -\bigcup_{i=1}^{k-1} B_i$.

If the process does not terminate in a finite number of steps, then the sequences $x^{(i)}$ and decoding sets $B_i$, $i = 1,2,...,u$, form the desired code. Thus assume that the process terminates after $t$ steps. (Conceivably $t = 0$). We will show $t \geq u$ by showing that $\epsilon \leq t e^{-t} + P\{ (X, Y) \notin A \} + P\{ X \notin F \}$. We proceed as follows.

Let

$$B = \bigcup_{j=1}^{t} B_j. \quad (If \quad t = 0, \quad take \quad B = \emptyset). \quad Then

$$P\{ (X, Y) \in A \} = \int_{(x,y) \in A} p(x,y) dx dy$$

$$= \int_{x} p(x) \int_{y \in A_x} p(y | x) dy dx$$

$$= \int_{x} p(x) \int_{y \in B \cap A_x} p(y | x) dy dx + \int_{x} p(x)$$

Algorithms
Ideals. Let A be a ring. Recall that an ideal a in A is a subset such that a is subgroup of A regarded as a group under addition; 
\( a \in A, r \in A \Rightarrow ra \in A \)

The ideal generated by a subset S of A is the intersection of all ideals A containing a ---- it is easy to verify that this is in fact an ideal, and that it consist of all finite sums of the form \( \sum r_s i \) with 
\( r_s \in A, s_i \in S \). When \( S = \{s_1, \ldots, s_m\} \), we shall write \( (s_1, \ldots, s_m) \) for the ideal it generates.

Let a and b be ideals in A. The set \( \{a+b \mid a \in a, b \in b\} \) is an ideal, denoted by \( a+b \).

The ideal generated by \( \{ab \mid a \in a, b \in b\} \) is denoted by \( ab \). Note that \( ab \subseteq a \cap b \). Clearly \( ab \) consists of all finite sums \( \sum a_i b_i \) with \( a_i \in a \) and \( b_i \in b \), and if \( a = (a_1, \ldots, a_m) \) and \( b = (b_1, \ldots, b_n) \), then \( ab = (a_1 b_1, \ldots, a_1 b_n, a_2 b_1, \ldots, a_2 b_n, \ldots, a_m b_1, \ldots, a_m b_n) \). Let \( a \) be an ideal of A. The set of cosets of \( a \) in A forms a ring \( A/ a \), and \( a \mapsto a + a \) is a homomorphism [1-161] \( \phi : A \mapsto A/a \). The map \( b \mapsto \phi^{-1}(b) \) is a one to one correspondence between the ideals of \( A/a \) and the ideals of A containing a.

An ideal \( p \) if prime if \( p \neq A \) and \( ab \in p \Rightarrow a \in p \) or \( b \in p \). Thus \( p \) is prime if and only if \( A/p \) is nonzero and has the property that \( ab = 0, \ b \neq 0 \) \( \Rightarrow a = 0, \) i.e., \( A/p \) is an integral domain. An ideal \( m \) is maximal if \( m \neq A \) and there does not exist an ideal \( n \) contained strictly between \( m \) and A. Thus \( m \) is maximal if and only if \( A/m \) has no proper nonzero ideals, and so is a field. Note that \( m \) maximal \( \Rightarrow m \) prime. The ideals of \( A \times B \) are all of the form \( a \times b \), with \( a \) and \( b \) ideals in \( A \) and \( B \). To see this, note that if \( c \) is an ideal in \( A \times B \) and \( (a,b) \in c \), then \( (a,0) = (a,b)(1,0) \in c \) and \( (0,b) = (a,b)(0,1) \in c \). This shows that \( c = a \times b \) with

\[
\begin{align*}
a &= \{a \mid (a,b) \in c \text{ some } b \in b\} \\
b &= \{b \mid (a,b) \in c \text{ some } a \in a\}
\end{align*}
\]

Let \( A \) be a ring. An \( A \)-algebra is a ring \( B \) together with a homomorphism \( i_b : A \mapsto B \). A homomorphism of \( A \)-algebra \( B \mapsto C \) is a homomorphism of rings \( \phi : B \mapsto C \) such that \( \phi(i_b(a)) = i_c(a) \) for all \( a \in A \). An \( A \)-algebra \( B \) is said to be finitely generated (or of finite-type over \( A \)) if there exist elements \( x_1, \ldots, x_n \in B \) such that every element of \( B \) can be expressed as a polynomial in the \( x_i \) with coefficients in \( i(A) \), i.e., such that the homomorphism \( A[x_1, \ldots, x_n] \mapsto B \) sending \( x_i \) to \( x_i \) is surjective. A ring \( A \)-homomorphism \( A \mapsto B \) is finite, and \( B \) is finitely generated as an \( A \)-module. Let \( k \) be a field, and let \( A \) be a \( k \)-algebra. If \( 1 \neq 0 \) in \( A \), then the map \( k \mapsto A \) is injective, we can identify \( k \) with its image, i.e., we can regard \( k \) as a subring of \( A \). If \( 1=0 \) in a ring \( R \), the \( R \) is the zero ring, i.e., \( R = \{0\} \).

Polynomial rings. Let \( k \) be a field. A monomial in \( X_1, \ldots, X_n \) is an expression of the form \( X_1^{a_1} \ldots X_n^{a_n}, \ a_j \in \mathbb{N} \). The total degree of the monomial is \( \sum a_j \). We sometimes abbreviate it by \( X^\alpha, \ \alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n \). The elements of the polynomial ring \( k[X_1, \ldots, X_n] \) are finite sums \( \sum c_{a_1} \ldots X_1^{a_1} \ldots X_n^{a_n}, \ c_{a_1} \ldots a_n \in k, \ a_j \in \mathbb{N}^n \). With the obvious notions of equality, addition and multiplication. Thus the monomials from basis for \( k[X_1, \ldots, X_n] \) as a \( k \)-vector space. The ring \( k[X_1, \ldots, X_n] \) is an integral domain, and the only units in it are the nonzero constant polynomials. A polynomial \( f(X_1, \ldots, X_n) \) is irreducible if it is nonconstant and has only the obvious factorizations,
i.e., \( f = gh \implies g \) or \( h \) is constant. **Division in**
\( k[X] \). The division algorithm allows us to divide a nonzero polynomial into another: let \( f \) and \( g \) be polynomials in \( k[X] \) with \( g \neq 0 \); then there exist unique polynomials \( q, r \in k[X] \) such that
\[ f = qg + r \]
with either \( r = 0 \) or \( \deg r < \deg g \). Moreover, there is an algorithm for deciding whether \( f \in (g) \), namely, find \( r \) and check whether it is zero. Moreover, the Euclidean algorithm allows to pass from finite set of generators for an ideal in \( k[X] \) to a single generator by successively replacing each pair of generators with their greatest common divisor. **(Pure) lexicographic ordering (lex).** Here monomials are ordered by lexicographic(dictionary) order. More precisely, let \( \alpha = (a_1, \ldots, a_n) \) and \( \beta = (b_1, \ldots, b_n) \) be two elements of \( \mathbb{Z}^n \); then \( \alpha > \beta \) and \( X^\alpha > X^\beta \) (lexicographic ordering) if, in the vector difference \( \alpha - \beta \in \mathbb{Z}^n \), the leftmost nonzero entry is positive. For example,
\[ XY^2 > Y^3Z^2, \quad X^3Y^2Z > X^3Y^2Z. \]
Note that this isn’t quite how the dictionary would order them: it would put \( XXXYZZZZ \) after \( XXXYZ \). **Graded reverse lexicographic order (grevlex).** Here monomials are ordered by total degree, with ties broken by reverse lexicographic ordering. Thus, \( \alpha > \beta \) if \( \sum a_i > \sum b_i \), or \( \sum a_i = \sum b_i \) and in \( \alpha - \beta \) the rightmost nonzero entry is negative. For example:
\[ X^4Y^4Z^2 > X^5Y^5Z^1, \quad XY^2Z^2 > X^4YZ^1, \quad X^5YZ > X^4YZ^2. \]
**Orderings on** \( k[X_1, \ldots, X_n] \). Fix an ordering on the monomials in \( k[X_1, \ldots, X_n] \). Then we can write an element \( f \) of \( k[X_1, \ldots, X_n] \) in a canonical fashion, by re-ordering its elements in decreasing order. For example, we would write
\[ f = 4XY^2Z + 4Z^2 - 5X^3 + 7X^2Z^2 \]
as
\[ f = -5X^3 + 7X^2Z^2 + 4XY^2Z + 4Z^2 \]
**lex**
or
\[ f = 4XY^2Z + 7X^2Z^2 - 5X^3 + 4Z^2 \]
grevlex
Let \( \sum a_iX^\alpha \in k[X_1, \ldots, X_n] \), in decreasing order:
\[ f = a_{\alpha_0}X^{\alpha_0} + a_1X^{\alpha_1} + \ldots, \quad \alpha_0 > \alpha_1 > \ldots, \quad \alpha_0 \neq 0 \]

Then we define.
- **The multidegree of** \( f \) to be \( \operatorname{multdeg}(f) = \alpha_0 \);
- **The leading coefficient of** \( f \) to be \( \operatorname{LC}(f) = a_{\alpha_0} \);
- **The leading monomial of** \( f \) to be \( \operatorname{LM}(f) = X^{\alpha_0} \);
- **The leading term of** \( f \) to be \( \operatorname{LT}(f) = a_{\alpha_0}X^{\alpha_0} \).

For the polynomial \( f = 4XY^2Z + \ldots, \) the multidegree is \( (1,2,1) \), the leading coefficient is 4, the leading monomial is \( XY^2Z \), and the leading term is \( 4XY^2Z \). **The division algorithm in** \( k[X_1, \ldots, X_n] \). Fix a monomial ordering in \( \mathbb{Z}^2 \). Suppose given a polynomial \( f \) and an ordered set \( \langle g_1, \ldots, g_s \rangle \) of polynomials; the division algorithm then constructs polynomials \( a_1, \ldots, a_s \) and \( r \) such that
\[ f = a_1g_1 + \ldots + a_sg_s + r \]
Where either \( r = 0 \) or no monomial in \( r \) is divisible by any of \( \langle LT(g_1), \ldots, LT(g_s) \rangle \). **Step 1:** If \( \langle LT(g_1) \rangle \mid LT(f) \).

divide \( g_1 \) into \( f \) to get
\[ f = a_1g_1 + h, \quad a_1 = \frac{LT(f)}{LT(g_1)} \in k[X_1, \ldots, X_n] \]
If \( \langle LT(g_1) \rangle \mid LT(h) \), repeat the process until \( f = a_1g_1 + f_1 \) (different \( a_1 \)) with \( LT(f_1) \) not divisible by \( LT(g_1) \). Now divide \( g_2 \) into \( f_1 \), and so on, until \( f = a_1g_1 + \ldots + a_sg_s + r_1 \) With

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\( LT(r_i) \) not divisible by any \( LT(g_1), \ldots, LT(g_s) \)

**Step 2:** Rewrite \( r_i = LT(r_i) + r_3 \), and repeat Step 1 with \( r_3 \) for \( f : \\
\begin{align*}
f &= a_1g_1 + \ldots + a_sg_s + LT(r_i) + r_3 \quad \text{(different} \\
a_i \text{'s)}
\end{align*}
\)

**Monomial ideals.** In general, an ideal \( a \) will contain a polynomial without containing the individual terms of the polynomial; for example, the ideal \( a = (Y^2 - X^3) \) contains \( Y^2 - X^3 \) but not \( Y^2 \) or \( X^3 \).

**DEFINITION 1.5.** An ideal \( a \) is monomial if \( \sum c_aX^a \in a \Rightarrow X^a \in a \) all \( a \) with \( c_a \neq 0 \).

**PROPOSITION 1.3.** Let \( a \) be a monomial ideal, and let \( A = \{ a \mid X^a \in a \} \). Then \( A \) satisfies the condition \( \alpha \in A, \beta \in \mathbb{Q}, a \Rightarrow \alpha + \beta \in \mathbb{Q} \)

And \( a \) is the \( k \)-subspace of \( k[X_1, \ldots, X_n] \) generated by \( X^a, \alpha \in A \). Conversely, if \( A \) is a subset of \( \mathbb{Q} \) satisfying \( \mathbb{Q} \), then the \( k \)-subspace \( a \) of \( k[X_1, \ldots, X_n] \) generated by \( \{ X^a \mid \alpha \in A \} \) is a monomial ideal.

**PROOF.** It is clear from its definition that a monomial ideal \( a \) is the \( k \)-subspace of \( k[X_1, \ldots, X_n] \) generated by the set of monomials it contains. If \( X^a \in a \) and \( X^a \in k[X_1, \ldots, X_n] \).

If a permutation is chosen uniformly and at random from the \( n! \) possible permutations in \( S_n \), then the counts \( C_j(n) \) of cycles of length \( j \) are independent random variables. The joint distribution of \( C_j(n) = (C_1(j,n), \ldots, C_n(j,n)) \) follows from Cauchy’s formula, and is given by

\[
P[C^{(n)} = c] = \frac{1}{n!} N(n,c) = \left[ \sum_j j^{n_j} = n \right] \prod_j \frac{1}{j!} c_j! \quad (1.1)
\]

for \( c \in \mathbb{Q}_+ \).

**Lemma 1.7.** For nonnegative integers \( m_1, \ldots, m_n \),

\[
E\left( \prod_{j=1}^n \left( C_j^{(n)} \right)^{m_j} \right) = \prod_{j=1}^n \left( \frac{1}{j!} \right)^{m_j} \sum_{j=1}^n j^{m_j} \leq n \quad (1.4)
\]

**Proof.** This can be established directly by exploiting cancellation of the form \( c_j^{m_j} / c_j' = 1 / (c_j - m_j) \)!

when \( c_j \geq m_j \), which occurs between the ingredients in Cauchy’s formula and the falling factorials in the moments. Write \( m = \sum j m_j \). Then, with the first sum indexed by \( c = (c_1, \ldots, c_n) \in \mathbb{Q}_+ \) and the last sum indexed by \( d = (d_1, \ldots, d_n) \in \mathbb{Q}_+ \) via the correspondence \( d_j = c_j - m_j, \) we have

\[
E\left( \prod_{j=1}^n \left( C_j^{(n)} \right)^{m_j} \right) = \prod_{j=1}^n P(C_j = c) \prod_{j=1}^n \left( \frac{1}{j!} \right)^{m_j}
\]

\[
= \sum_{c_1 \geq \ldots \geq c_n \text{ for all } j} \prod_{j=1}^n \left( c_j^{m_j} \right) \prod_{j=1}^n \left( \frac{1}{j!} \right)^{m_j}
\]

\[
= \prod_{j=1}^n \frac{1}{j!} \sum_{d_1 \leq \ldots \leq d_n} \prod_{j=1}^n \left( d_j^{n-m} \right) \prod_{j=1}^n \left( \frac{1}{j!} \right)^{m_j}
\]

This last sum simplifies to the indicator \( 1 \quad (n \leq n) \), corresponding to the fact that if \( n - m \geq 0 \), then \( d_j = 0 \) for \( j > n - m \), and a random permutation in \( S_{n-m} \) must have some cycle structure \((d_1, \ldots, d_{n-m}) \). The moments of \( C_j^{(n)} \) follow immediately as

\[
E\left( C_j^{(n)} \right)^{1} = j^{-1} \{ j r \leq n \} \quad (1.2)
\]

We note for future reference that (1.4) can also be written in the form

\[
E\left( \prod_{j=1}^n \left( C_j^{(n)} \right)^{m_j} \right) = E\left( \prod_{j=1}^n Z_j^{m_j} \right) \sum_{j=1}^n j^{m_j} \leq n \quad (1.3)
\]

Where the \( Z_j \) are independent Poisson-distribution random variables that satisfy

\[
E(Z_j) = 1 / j
\]

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The marginal distribution of cycle counts provides a formula for the joint distribution of the cycle counts \( C_j^n \), we find the distribution of \( C_j^n \) using a combinatorial approach combined with the inclusion-exclusion formula.

**Lemma 1.8.** For \( 1 \leq j \leq n \),
\[
P(C_j^n = k) = \frac{L_{j+1}^{n-j} + 1}{k!} \sum_{i=0}^{k} (-1)^i \frac{j!}{i!} \quad (1.1)
\]

**Proof.** Consider the set \( I \) of all possible cycles of length \( j \), formed with elements chosen from \( \{1, 2, \ldots, n\} \), so that \(|I| = n^j/j!\). For each \( \alpha \in I \), consider the “property” \( G_\alpha \) of having \( \alpha \); that is, \( G_\alpha \) is the set of permutations \( \pi \in S_n \) such that \( \alpha \) is one of the cycles of \( \pi \). We then have \( |G_\alpha| = (n-j)! \), since the elements of \( \{1, 2, \ldots, n\} \) not in \( \alpha \) must be permuted among themselves. To use the inclusion-exclusion formula we need to calculate the term \( S_i \), which is the sum of the probabilities of the \( r \)-fold intersection of properties, summing over all sets of \( r \) distinct properties. There are two cases to consider. If the \( r \) properties are indexed by \( r \) cycles having no elements in common, then the intersection specifies how \( rj \) elements are moved by the permutation, and there are \((n-rj)!/(rj!)\) such permutations. For the other case, some two distinct properties name some element in common, so no permutation can have both these properties, and the \( r \)-fold intersection is empty. Thus \( S_i = (n-rj)!/rj! \) is the number of permutations having exactly \( k \) properties is
\[
\sum_{j \geq 0} (-1)^j \binom{k+j}{l} S_{k+l}. 
\]

Which simplifies to (1.1) Returning to the original hat-check problem, we substitute \( j=1 \) in (1.1) to obtain the distribution of the number of fixed points of a random permutation. For \( k = 0, 1, \ldots, n \),
\[
P(C_1^n = k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \frac{1}{i!}, \quad (1.2)
\]

and the moments of \( C_1^n \) follow from (1.2) with \( j = 1 \). In particular, for \( n \geq 2 \), the mean and variance of \( C_1^n \) are both equal to 1. The joint distribution of \( (C_1^n, \ldots, C_b^n) \) for any \( 1 \leq b \leq n \) has an expression similar to (1.7); this too can be derived by inclusion-exclusion. For any \( c = (c_1, c_2, \ldots, c_b) \in \mathbb{N}_b^b \) with \( m = \sum c_j \),
\[
P(C_1^n, \ldots, C_b^n = c) = \prod_{j=1}^{b} \left( \frac{1}{c_j!} \sum_{l=0}^{c_j} (-1)^{c_j-l} \prod_{k=1}^{l} \frac{1}{k!} \right) = \sum_{c_1+\cdots+c_b=n} \prod_{j=1}^{b} \frac{1}{c_j!} \sum_{l=0}^{c_j} (-1)^{c_j-l} \prod_{k=1}^{l} \frac{1}{k!} \left( \frac{1}{c_j!} \right) (1.3)
\]

The joint moments of the first \( b \) counts \( C_1^n, \ldots, C_b^n \) can be obtained directly from (1.2) and (1.3) by setting \( m_{b+1} = \ldots = m_n = 0 \)

The limit distribution of cycle counts

It follows immediately from Lemma 1.2 that for each fixed \( j \), as \( n \to \infty \),
\[
P(C_j^n = k) \to \frac{k^{-j} e^{-1/j}}{k!}, \quad k = 0, 1, 2, \ldots,
\]

So that \( C_j^n \) converges in distribution to a random variable \( Z_j \) having a Poisson distribution with mean \( 1/j \); we use the notation \( C_j^n \to d Z_j \) to describe this. In fact, the limit random variables are independent.

**Theorem 1.6** The process of cycle counts converges in distribution to a Poisson process of \( \hat{d} \) with intensity \( j^{-1} \). That is, as \( n \to \infty \),
\[
(C_1^n, C_2^n, \ldots) \to_d (Z_1, Z_2, \ldots) \quad (1.1)
\]
Where the \( Z_j, j = 1, 2, \ldots, \) are independent Poisson-distributed random variables with
\[
E(Z_j) = \frac{1}{j}
\]

**Proof.** To establish the converges in distribution one shows that for each fixed \( b \geq 1, \) as \( n \to \infty, \)
\[
P[(C_1^{(n)}, \ldots, C_b^{(n)}) = c] \to P[(Z_1, \ldots, Z_b) = c]
\]

**Error rates**

The proof of Theorem says nothing about the rate of convergence. Elementary analysis can be used to estimate this rate when \( b = 1. \) Using properties of alternating series with decreasing terms, for \( k = 0, 1, \ldots, n, \)
\[
\frac{1}{k!} \left( \frac{1}{(n-k+1)!} - \frac{1}{(n-k+2)!} \right) \leq \left| P[C_i^{(n)} = k] - P[Z_i = k] \right| \\
\leq \frac{1}{k!(n-k+1)!}
\]

It follows that
\[
\frac{2^{2n}}{(n+1)!} \leq \sum_{i=0}^{n} \frac{1}{i!} \leq \frac{2^{2n} - 1}{(n+1)!}
\]

Since
\[
P[Z_i > n] = e^{-1} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots \right) < \frac{1}{(n+1)!}
\]

We see from (1.11) that the total variation distance between the distribution \( L(C_i^{(n)}) \) of \( C_i^{(n)} \) and the distribution \( L(Z_i) \) of \( Z_i \)

Establish the asymptotics of \( P[A_g(C^{(n)})] \) under conditions \((A_0)\) and \((B_0),\)

\[
A_g(C^{(n)}) = \bigcap_{l \leq n} \bigcap_{i+L \in L_i} \left\{ C_j^{(n)} = 0 \right\},
\]

and \( \xi_i = r_i / r_d - 1 = O(\xi^k) \) as \( i \to \infty, \) for some \( g > 0. \) We start with the expression
\[
P[A_g(C^{(n)})] = \frac{P[T_{0n}(Z) = n]}{P[T_{0n}(Z) = n]}
\]

\[
\prod_{l \leq n} \left\{ 1 - \frac{\xi}{ir_i}(1 + E_{l_0}) \right\}
\]

\[
P[T_{0n}(Z) = n] = \frac{\theta d}{n} \exp \left\{ \sum_{i=1}^{\infty} \left[ \log(1 + i^{-1} d) - i^{-1} d \right] \right\}
\]

\[
\{ 1 + O(n^{-1} \varphi_{1,2,7}(n)) \} \quad (1.2)
\]

and
\[
P[T_{0n}(Z) = n] = \frac{\theta d}{n} \exp \left\{ \sum_{i=1}^{\infty} \left[ \log(1 + i^{-1} d) - i^{-1} d \right] \right\}
\]

\[
\{ 1 + O(n^{-1} \varphi_{1,2,7}(n)) \} \quad (1.3)
\]

Where \( \varphi_{1,2,7}(n) \) refers to the quantity derived from \( Z. \) It thus follows that

\[
P[A_g(C^{(n)})] \leq Kn^{-\theta(1-d)}
\]

for a constant \( K, \) depending on \( Z \) and the \( r_i \) and computable explicitly from (1.1) – (1.3), if Conditions \((A_0)\) and \((B_0)\) are satisfied and if \( \xi_i = O(\xi^k) \) from some \( g > 0, \) since, under these circumstances, both \( n^{-1} \varphi_{1,2,7}(n) \) and \( n^{-1} \varphi_{1,2,7}(n) \) tend to zero as \( n \to \infty. \) In particular, for polynomials and square free polynomials, the relative error in this asymptotic approximation is of order \( n^{-1} \) if \( g > 1. \)

For \( 0 \leq b \leq n / 8 \) and \( n \geq n_0, \) with \( n_0 \)

\[
d_{TV}(L(C[1, b]), L(Z[1, b])) \\
\leq d_{TV}(L(C[1, b]), L(Z[1, b]))
\]

\[
\leq \varepsilon_{[7,7]}(n, b),
\]

Where \( \varepsilon_{[7,7]}(n, b) = O(b / n) \) under Conditions \((A_0), (D_1)\) and \((B_1)\) Since, by the Conditioning Relation,

\[
L(C[1, b] | T_{0n}(C) = l) = L(Z[1, b] | T_{0n}(Z) = l)
\]

It follows by direct calculation that
\[ d_{Py}(L(C[1,b]), L(Z[1,b])) = d_{Py}(L(T_{0b}(C)), L(T_{0b}(Z))) = \max_{r \in A} \sum_{r \in A} P[T_{0b}(Z) = r] \left\{ 1 - \frac{P[T_{0b}(Z) = n - r]}{P[T_{0b}(Z) = n]} \right\} \]

(1.4)

Suppressing the argument \( Z \) from now on, we thus obtain

\[ d_{Py}(L(C[1,b]), L(Z[1,b])) = \sum_{r \in A} P[T_{0b} = r] \left\{ 1 - \frac{P[T_{0b} = n - r]}{P[T_{0b} = n]} \right\} \]

\[ \leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r = 0}^{[n/2]} P[T_{0b} = r] \]

\[ \times \left\{ \sum_{s = 0}^{[n/2]} P[T_{0b} = s] \left\{ P[T_{bn} = n - s] - P[T_{bn} = n - r] \right\} \right\} \]

\[ \leq \sum_{r > n/2} P[T_{0b} = r] + \sum_{r = 0}^{[n/2]} P[T_{0b} = r] \]

\[ \times \sum_{s = 0}^{[n/2]} P[T_{0b} = s] \frac{P[T_{bn} = n - s] - P[T_{bn} = n - r]}{P[T_{bn} = n]} \]

\[ + \sum_{s = 0}^{[n/2]} P[T_{0b} = r] \sum_{i = [n/2] + 1} P[T = s] P[T_{bn} = n - s] \frac{P[T_{bn} = n]}{P[T_{bn} = n]} \]

\[ \frac{3n}{\theta P_{\theta}[0,1]} 4n^{-2} \phi_{[10.8]}(n) \sum_{r = 0}^{[n/2]} P[T_{0b} = r] \sum_{s = 0}^{[n/2]} P[T_{0b} = s] \frac{1}{2} |r - s| \]

\[ \leq \frac{12 \phi_{[10.8]}(n) E_{T_{0b}}}{\theta P_{\theta}[0,1]} \]

Hence we may take

\[ \epsilon_{[7.7]}(n,b) = 2n^{-1} E_{T_{0b}}(Z) \left\{ 1 + \frac{6 \phi_{[10.8]}(n)}{\theta P_{\theta}[0,1]} \right\} P_{\theta}^{1/2} \]

(1.5)

Required order under Conditions \((A_1), (D_1)\) and \((B_{11})\), if \( S(\infty) < \infty \). If not, \( \phi_{[10.8]}(n) \) can be replaced by \( \phi_{[10.11]}(n) \) in the above, which has the required order, without the restriction on the \( r \), implied by \( S(\infty) < \infty \). Examining the Conditions \((A_1), (D_1)\) and \((B_{11})\), it is perhaps surprising to find that \((B_{11})\) is required instead of just \((B_{01})\); that is, that we should need

\[ \sum_{i \geq 2} \epsilon_{ii} = O(i^{-a}) \]

for \( \epsilon_{ii} \) to be of order \( O(b/n) \). This makes it possible to replace condition \((A_1)\) by the weaker pair of conditions \((A_1)\) and \((D_1)\) in the eventual assumptions needed for \( \epsilon_{[7.7]}(n,b) \) to be of order \( O(b/n) \); the decay rate requirement[1-161] of order \( i^{-1-\gamma} \) is shifted from \( \epsilon_{ii} \) itself to its first difference. This is needed to obtain the right approximation error for the random mappings example. However, since all the classical applications make far more stringent assumptions about the \( \epsilon_{ii} \), \( i \geq 2 \), than are made in \((B_{11})\). The critical point of the proof is seen where the initial estimate of the difference

\[ P[T_{bn}^{(m)} = n] - P[T_{bn}^{(m)} = s + 1] \]

of the difference \( \epsilon_{[10.10]}(n) \), which should be small, contains a far tail element from \( n_1 \) of the form \( \phi_{[10.8]}(n) + u_{[1]}(n) \), which is only small if \( a_1 > 1 \), being otherwise of order \( O(n^{-a_1+\delta}) \) for any \( \delta > 0 \), since \( a_2 > 1 \) in any case assumed. For \( s \geq n/2 \), this gives rise to a
contribution of order $O(n^{-\alpha_0+\delta})$ in the estimate of the difference $P[T_{bn} = s] - P[T_{bn} = s+1]$, which, in the remainder of the proof, is translated into a contribution of order $O(m^{-\alpha_0+\delta})$ for differences of the form $P[T_{bn} = s] - P[T_{bn} = s+1]$, finally leading to a contribution of order $bn^{-\alpha_0+\delta}$ for any $\delta > 0$ in $e_{[7,7]}(n, b)$. Some improvement would seem to be possible, defining the function $g$ by $g(w) = 1_{\{w = x\}} - 1_{\{w = x + \epsilon\}}$, differences that are of the form $P[T_{bn} = s] - P[T_{bn} = s+t]$ can be directly estimated, at a cost of only a single contribution of the form $\phi^b(n) + \phi^t(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form $\beta_{(1,2)}^b(n) + \beta_{(1,2)}^t(n)$. Then, iterating the cycle, in which one estimate of a difference in point probabilities is improved to an estimate of smaller order, a bound of the form $\sum_{r=0}^{\infty} P[T_{ob} = r] \left\{ 1 - \frac{P[T_{on} = n-r]}{P[T_{on} = n]} \right\} - \sum_{r=\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} P[T_{ob} = r] \right\} \sum_{r=0}^{n} P[T_{ob} = r] \left( P[T_{on} = n-s] - P[T_{on} = n-r] \right) \leq 4n^{-2}E^2_{ob} + \left( \max_{n/2 \geq r \geq n} P[T_{ob} = s] / P[T_{on} = n] \right) + P[T_{ob} > n/2] \leq 8n^{-2}E^2_{ob} + \frac{3e_{[0.5,2]}(n/2, b)}{\theta_{P_r}([0,1])}, \quad (1.1)

3. Implementation

After several weeks of onerous designing, we finally have a working implementation of our system. Since we allow wide-area networks to emulate metamorphic methodologies without the development of flip-flop gates, designing the homegrown database was relatively straightforward [10]. Similarly, SopE requires root access in order to measure homogeneous algorithms. Overall, our algorithm adds only modest overhead and complexity to existing multimodal frameworks.

4. Evaluation

Evaluating a system as novel as ours proved more arduous than with previous systems. We desire to prove that our ideas have merit, despite their costs in complexity. Our overall performance analysis seeks to prove three hypotheses: (1) that expected bandwidth is a bad way to measure average popularity of e-commerce [11]; (2) that public-private key pairs no longer impact performance; and finally (3) that vacuum tubes have actually shown exaggerated block size over time. We are grateful for disjoint von Neumann machines; without them, we could not optimize for usability simultaneously with usability. An astute reader would now infer that for obvious reasons, we have decided not to explore a methodology’s API. Along these same lines, only with the benefit of our system’s mobile ABI might we optimize for security at the cost of response time. Our performance analysis will show that reducing the USB key space of modular configurations is crucial to our results.
4.1 Hardware and Software Configuration

![Figure 2: The median work factor of Sope, compared with the other systems.](image)

A well-tuned network setup holds the key to an useful evaluation method. We executed a simulation on UC Berkeley's XBox network to quantify the work of Italian complexity theorist I. Takahashi. We struggled to amass the necessary RISC processors. To start off with, we added some tape drive space to the KGB's mobile telephones. The 2GHz Intel 386s described here explain our conventional results. We removed 8Gb/s of Ethernet access from our interposable cluster. Continuing with this rationale, we doubled the effective floppy disk throughput of our mobile telephones to investigate configurations. We struggled to amass the necessary 5.25" floppy drives. Along these same lines, we removed more NV-RAM from our network.

![Figure 3: Note that complexity grows as distance decreases - a phenomenon worth analyzing in its own right.](image)

We ran our framework on commodity operating systems, such as OpenBSD and Ultrix Version 5.1.2. we added support for our application as an embedded application. Our experiments soon proved that exokernelizing our local-area networks was more effective than distributing them, as previous work suggested. On a similar note, we made all of our software is available under a Microsoft-style license.

![Figure 4: These results were obtained by Zheng et al. [12]; we reproduce them here for clarity.](image)
4.2 Experimental Results

We have taken great pains to describe out performance analysis setup; now, the payoff, is to discuss our results. We ran four novel experiments: (1) we measured E-mail and E-mail throughput on our 10-node overlay network; (2) we measured RAM throughput as a function of NV-RAM space on an Apple Newton; (3) we deployed 04 IBM PC Juniors across the Internet-2 network, and tested our semaphores accordingly; and (4) we measured DNS and instant messenger performance on our desktop machines.

Now for the climactic analysis of the second half of our experiments. The curve in Figure 2 should look familiar; it is better known as $G(n) = n$. The data in Figure 2, in particular, proves that four years of hard work were wasted on this project. Similarly, note the heavy tail on the CDF in Figure 2, exhibiting weakened expected seek time.

Lastly, we discuss experiments (3) and (4) enumerated above. Note that Figure 2 shows the 10th-percentile and not average partitioned ROM speed. Operator error alone cannot account for these results. Furthermore, bugs in our system caused the unstable behavior throughout the experiments [13].

5. Related Work

In designing Sope, we drew on prior work from a number of distinct areas. The choice of Markov models in [14] differs from ours in that we simulate only key configurations in our heuristic [15]. Despite the fact that John Backus also motivated this solution, we harnessed it independently and simultaneously [16,17,18]. Our design avoids this overhead. Jackson and Bose et al. [6,19] described the first known instance of Boolean logic [20].

Our solution is related to research into the Turing machine [18], compact theory, and "smart" algorithms [21]. Similarly, unlike many existing solutions, we do not attempt to observe or emulate voice-over-IP [22,8]. Further, Sasaki and Nehru developed a similar system, nevertheless we showed that Sope is maximally efficient [23]. These systems typically require that expert systems and superblocks are often incompatible, and we disproved here that this, indeed, is the case.

While we know of no other studies on write-ahead logging, several efforts have been made to synthesize RAID [24] [2,5,13]. The choice of checksums in [25] differs from ours in that we simulate only key symmetries in our approach [26]. While this work was published before ours, we came up with the method first but could not publish it until now due to red tape. Continuing with this rationale, a recent unpublished undergraduate dissertation [27] introduced a similar idea for sensor networks [28]. A litany of prior work supports our use of autonomous methodologies. In the end, note that Sope is built on the refinement of compilers; clearly, our solution runs in $\Omega(n^2)$ time [9,29,30,31].

6. Conclusion

In conclusion, we concentrated our efforts on demonstrating that the well-known wearable algorithm for the understanding of multicast algorithms by U. Wilson et al. [32] is maximally efficient. Furthermore, one potentially minimal flaw of Sope is that it cannot manage Lamport clocks; we
plan to address this in future work. We disconfirmed that though systems and reinforcement learning can collude to fulfill this objective, the acclaimed extensible algorithm for the improvement of randomized algorithms by Davis et al. runs in O(log log n) time. Sope has set a precedent for courseware, and we expect that physicists will improve Sope for years to come. One potentially profound disadvantage of Sope is that it cannot construct introspective modalities; we plan to address this in future work.

References

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